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The limit functions of a random iteration system ☆

Ji Zhou [a,*] and Mao-Kang Luo [b]

[a] *Department of Mathematics, Sichuan Normal University, Chengdu 610066, China*
[b] *Department of Mathematics, Sichuan University, Chengdu 610064, China*

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Abstract

This paper discusses the limit functions of a random iteration system formed by finitely many rational functions. Applying these results we prove that a hyperbolic iteration system has no wandering domain and that its limit functions are constant. Finally the continuity on its Julia set is considered.

Keywords: Fatou sets; Julia sets; Normal family

1. Introduction

Let $\mathcal{R} = \{R_1, R_2, \ldots, R_M\}$ be a set of rational functions with degree more than one. Suppose that $Y = \{1, 2, \ldots, M\}$ and $\Sigma_M = \prod_0^\infty Y$. For each orbit $\sigma = (j_1, j_2, \ldots, j_n, \ldots) \in \Sigma_M$, we define

$$W_\sigma^0(z) = z, \qquad W_\sigma^n(z) = R_{j_n} \circ R_{j_{n-1}} \circ \cdots \circ R_{j_1}(z),$$

and its inverse $W_\sigma^{-n}(z)$,

$$W_\sigma^{-n}(z) = \left(W_\sigma^n\right)^{-1}(z) = R_{j_1}^{-1} \circ R_{j_2}^{-1} \circ \cdots \circ R_{j_n}^{-1}(z), \quad n = 1, 2, \ldots.$$

It is known [4,5] that the Julia set of the random iteration system formed by $\mathcal{R}$ is the closure of union of set of non-normality of the sequences $\{W_\sigma^n(z)\}$ for all orbits σ in Σ_M, denoted by $J(\mathcal{R})$. The complement of $J(\mathcal{R})$ is called the Fatou set of the random iteration

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* Corresponding author.
E-mail address: jzhou1@163.net (J. Zhou).

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system formed by $\mathcal{R}$, denoted by $F(\mathcal{R})$. Further, each component of $F(\mathcal{R})$ is called a Fatou component. If R is a rational function, $J(R)$ and $F(R)$ denote the Julia set and Fatou set of R, respectively.

In the classical case (the iteration of one rational function) both the Julia set and Fatou set are completely invariant. For the random iteration system formed by $\mathcal{R}$ these sets are, however, not necessarily completely invariant, and so study of it becomes more complicated. Some properties of the Julia set similar to the classical case (see [4,5]) have been obtained. But few researches about the Fatou set of the random iteration system have been made. It is known that the limit functions play an important role in study of the iteration of one rational function; in fact, the forward invariant Fatou components may be classified by the limit functions [1,2]. In this paper we first develop some properties of the limit functions of the random iteration system and then, with the aid of these results, investigate hyperbolic iteration systems and finally consider continuity of Julia sets of the random iteration systems.

Throughout this paper we denote by S the one-sided shift from Σ_M onto itself. Hence $S(\sigma) = (j_2, j_3, \ldots, j_n, \ldots)$ when $\sigma = (j_1, j_2, j_3, \ldots, j_n, \ldots)$.

2. Limit functions

We begin this section with the following definition.

Definition 1. Suppose that U is a component of $F(\mathcal{R})$, a function φ is called a limit function in U if there are a sequence $\{n_k\}$ of positive integers and $\sigma \in \Sigma_M$, such that $W_\sigma^{n_k} \to \varphi$ locally uniformly in U as $k \to \infty$. To specify the orbit explicitly, sometimes φ is called a limit function in U for the orbit σ.

The following cases do not occur in the iteration of a rational functions and show that the limit functions of the random iteration system are more complicated than these of the classical case.

Example 1. It is easily seen that, if $R_1 = z^2$ and $R_2 = z^2 + c$ for c small enough, there is a Fatou component V containing the origin with $R_i(V) \subset V$, $i = 1, 2$, on which there are the constant limit functions 0 and $1/4 - \sqrt{1/4 - c}$, which are the fixed points of R_1 and R_2, respectively, for the different orbits. Further, take $\sigma_1 = (1, 2, 1, 2, \ldots)$, then there are two constant limit functions a and $R_1(a)$ in V for the orbit σ_1, where a is the root of $R_2 \circ R_1(z) = z^4 + c = z$ in V, since for σ_1, $W_{\sigma_1}^{2m} \to a$ and $W_{\sigma_1}^{2m+1} \to R_1(a)$ locally uniformly in V as $m \to \infty$.

Example 2. Let R_1 and R_2 with $R_1(0) = R_2(0) = 0$ be rational functions satisfying, for some mapping G that is injective in some neighborhood Ω of the origin O and that fixes O, $G^{-1}R_1G(z) = \alpha z^2$ with $\alpha > 0$ and $G^{-1}R_2G(z) = e^{i\theta}z$ on Ω. According to the classical theory, we can choose suitable α and θ such that there exists a neighborhood V of O satisfying $G^{-1}R_jG(V) \subset V$, $j = 1, 2$. Now R_2 possesses a Siegel disk U containing O and hence $R_j(V') \subset V'$, $j = 1, 2$, where $V' = G(V)$. It follows that there is a component

F_0 of the Fatou set of the random iteration system formed by $\{R_1, R_2\}$ such that O lies in F_0. Clearly, there are a constant limit function 0 and a non-constant limit function in F_0 for different orbits.

Let $\sigma \in \Sigma_M$,

$$P(\sigma) = \bigcup_{n>0} \{\text{critical values of } W_\sigma^n\},$$

and

$$P(\mathcal{R}) = \bigcup_{\sigma \in \Sigma_M} \overline{P(\sigma)}. \tag{1}$$

In the classical case each constant limit function attracts a forward orbit of some critical point. For the random iteration system we have

Theorem 1. *If there is a constant limit function φ in a component U of $F(\mathcal{R})$ with value ζ, then ζ lies in $P(\mathcal{R})$.*

Proof. Suppose to the contrary that for some $r > 0$, $D = \{|z - \zeta| < r\}$ does not meet $P(\mathcal{R})$. By the definition there are $\sigma \in \Sigma_M$ and a set N of positive integers such that $W_\sigma^n(z)$ converges to ζ locally uniformly in U as $n \to \infty$ in N. Take $z' \in U$, and for large $n \in N$ with $W_\sigma^n(z') \in D$, we have a single-value analytic branch W_n of W_σ^{-n} in D satisfying $W_n(W_\sigma^n(z')) = z'$, and that $W_n(D)$ is disjoint from the critical points of $W_\sigma^n(z)$. This implies that $W_n(z)$, $n \in N$, is normal in D. Assume that $W_n \to \psi$ in D locally uniformly when $n \to \infty$ in some subset of N, also say N. Hence we obtain

$$z' = W_n\big(W_\sigma^n(z')\big) \to \psi(\zeta)$$

as $n \to \infty$ in N. Now taking another point z_1' in U, we can also obtain that $z_1' = \psi(\zeta)$ and hence $z' = z_1'$. This is a requiring contradiction and the proof is complete. □

Theorem 2. *If there is a non-constant limit function φ in the Fatou component U, then the identity map is a limit function and some function $R_i \in \mathcal{R}$ is injective in some component of $F(\mathcal{R})$.*

Proof. If the limit function φ in some component U of $F(\mathcal{R})$ is non-constant, noting that [5] $R_i(F(\mathcal{R})) \subset F(\mathcal{R})$ and $R_i^{-1}(J(\mathcal{R})) \subset J(\mathcal{R})$ for each $R_i \in \mathcal{R}$, we have

$$\varphi(U) \subset F(\mathcal{R}),$$

and there are a sequence $\{n_i\}$ of positive integers and $\sigma = (j_1, j_2, \ldots, j_n, \ldots) \in \Sigma_M$ such that $W_\sigma^{n_i} \to \varphi$ locally uniformly in U as $i \to \infty$. Hence there are a positive integer ℓ and a component U of $F(\mathcal{R})$ such that $W_\sigma^{n_i}(U) \subset V$ when $i > \ell$. Thus now we assume that the sequence $\{n_i\}$ and the orbit σ satisfy that $W_\sigma^{n_i}(U) \subset U$ and $W_\sigma^{n_i} \to \varphi$ locally uniformly in U as $i \to \infty$.

By passing to a subsequence of $\{n_i\}$ and relabeling, we may assume that $m_i = n_i - n_{i-1} \to \infty$, as $i \to \infty$. Write

$$G_{m_i}(z) = W_{S^{n_{i-1}}(\sigma)}^{m_i}(z) = R_{j_{n_i}} \circ \cdots \circ R_{j_{n_{i-1}+1}}(z).$$

Now G_{m_i} is normal in U since $G_{m_i}(U) \subset U$, and so there is a function ψ such that $G_{m_i} \to \psi$ locally uniformly in U as $i \to \infty$ in some set N of positive integers. Hence we have

$$\psi\varphi(z) = \lim G_{m_i}\bigl(W_\sigma^{n_{i-1}}(z)\bigr) = \lim W_\sigma^{n_i}(z) = \varphi(z), \quad z \in U,$$

as $i \to \infty$ in the set N. Since φ is a non-constant function, ψ must be the identity map.

Since $\mathcal{R}$ is a finite set of rational functions, we may take a subsequence of G_{m_i}, also say G_{m_i}, such that $W^1_{S^{n_i}(\sigma)}(z) = R_{j_n+1}$ is the same, say f. Next we show that f is injective in U. If $f(a) = f(b)$, then

$$\begin{aligned} G_{m_i}(a) &= W^{m_i}_{S^{n_{i-1}}(\sigma)}(a) = W^{m_i-1}_{S^{n_{i-1}+1}(\sigma)}\bigl(f(a)\bigr) = W^{m_i-1}_{S^{n_{i-1}+1}(\sigma)}\bigl(f(b)\bigr) = W^{m_i}_{S^{n_{i-1}}(\sigma)}(b) \\ &= G_{m_i}(b), \end{aligned}$$

and letting $i \to \infty$ in the set N, we get $a = b$. The proof is complete. □

Appealing to the above theorem, it easy to see that if U is a component of $F(\mathcal{R})$, and if there is a non-constant limit function φ on U, then at least one rational function R_j in $\mathcal{R}$ such that R_j possesses Siegel disks or Herman rings.

3. Hyperbolic iteration systems

Now introduce a hyperbolic iteration system as follows:

Definition 2. Let $\sigma \in \Sigma_M$ and $P(\mathcal{R})$ as in (1). $\mathcal{R}$ is hyperbolic if $P(\mathcal{R}) \subset F(\mathcal{R})$.

Clearly, if $\mathcal{R} = \{R\}$, that is to say, only one element is in $\mathcal{R}$, the fact that $\mathcal{R}$ is hyperbolic implies that R is hyperbolic in the common sense. It is known that if R is hyperbolic, then its Julia set $J(R)$ has no interior points. However, if $\mathcal{R}$ contains at least two elements and is hyperbolic, it is possible that there exists an interior point in its Julia set $J(\mathcal{R})$. For example, assume that $R_1 = z^2$, $R_2 = (1/2)z^2$. The system $\{R_1, R_2\}$ is hyperbolic since 0 and ∞ are both its fixed points and critical points, and its Julia set is $J = \{1 \leqslant |z| \leqslant 2\}$.

For each component of the Fatou set $F(\mathcal{R})$, it is not necessarily onto another by $R_i \in \mathcal{R}$ [5], hence we should deal carefully with defining a wandering domain of $\mathcal{R}$. Let U be a component of $F(\mathcal{R})$ and U_σ^n denotes the component of $F(\mathcal{R})$ containing $W_\sigma^n(U)$. We give

Definition 3. A component U of $F(\mathcal{R})$ is wandering if there are a $\sigma \in \Sigma_M$ and a sequence $\{n_j\}$ of positive integers such that $U_\sigma^{n_j} \cap U_\sigma^{n_k} = \emptyset$, $j \neq k$.

By the definition, if $\mathcal{R} = \{R\}$ and $\mathcal{R}$ has a wandering component U of the Fatou set $F(\mathcal{R})$, then U is also a wandering component of the Fatou set $F(R)$ in the usual sense. Indeed, if for some sequence of positive integers $\{n_k\}$ and some component U of Fatou set $F(R)$ of R, we have $R^{n_k}(U) \neq R^{n_j}(U)$, $k \neq j$, then for any positive integer m, l with $m \neq l$, it must be true that $R^m(U) \neq R^l(U)$.

It is well-known that every component of the Fatou set of a rational function is eventually periodic. When $\mathcal{R}$ is hyperbolic, we have the following result similar to the classical case.

Theorem 3. *If $\mathcal{R}$ is hyperbolic, then each component of $F(\mathcal{R})$ is non-wandering.*

Proof. Since if $\mathcal{R}$ has a wandering component U of $F(\mathcal{R})$, then there exist $\sigma \in \Sigma_M$ and a sequence $\{n_i\}$ of positive integers such that $U_\sigma^{n_j} \neq U_\sigma^{n_i}$ when $j \neq i$. Notice that $W_\sigma^n(z)$ forms a normal family in U, hence there exists a subsequence of $\{n_i\}$, also say $\{n_i\}$, such that $W_\sigma^{n_i}(z)$ converges to some function $g(z)$ locally uniformly in U. Obviously $g(z)$ is a constant function in U, since otherwise $g(U)$ would contain some component, say V, of $F(\mathcal{R})$ such that for large n_i, all $W_\sigma^{n_i}(U)$ would lie in V; this contradicts the fact that U is wandering. Since $\mathcal{R}$ is hyperbolic, in view of Theorem 1, $g(z)$ must be a constant function with value in $F(\mathcal{R})$, which again contradicts the fact that U is wandering. The proof is complete. □

For limit functions of the hyperbolic iteration system, we obtain

Theorem 4. *If R is hyperbolic, then all limit functions are constant with values in $F(\mathcal{R})$.*

Proof. By Theorem 2 it follows that if a limit function were non-constant, then there would exist some $R_i \in \mathcal{R}$ such that R_i were injective in some component of $F(\mathcal{R})$. According to the classical results, we see that R_i must possess a Siegel disk or Herman ring. The closure of its postcritical points would meet the Julia set $J(R_i)$, and $J(\mathcal{R}) \cap P(\mathcal{R}) \neq \emptyset$ since $J(R_i) \subset J(\mathcal{R})$ and the closure of postcritical points of R_i belongs to $P(\mathcal{R})$; it is a contradiction. Thus no limit function is non-constant. Now Theorem 1 and the assumption imply that if a limit function is constant, then it assumes value in $F(\mathcal{R})$. This completes the proof. □

4. Continuity of Julia sets

Let W_i, $i = 1, 2, \ldots, M$, be complex manifolds, and for each i, let $R_{w_i}(z) = R(w_i, z)$: $W_i \times \overline{\mathbf{C}} \mapsto \overline{\mathbf{C}}$ be a holomorphic function and for every $w_i \in W_i$, $R_{w_i}(z) = R(w_i, z)$ be a rational function. Further we assume that the degree of $R_{w_i}(z)$ is at least two. Then $\mathcal{R}_w \triangleq \mathcal{R}(w_1, w_2, \ldots, w_M) \triangleq \{R_{w_1}, R_{w_2}, \ldots, R_{w_M}\}$, $w \in \prod_{i=1}^M W_i$, is a holomorphic family of rational functions.

If R is a rational function, the continuity of $J(R)$ under the Hausdorff metric on the collection of all compact subsets of $\overline{\mathbf{C}}$ has be considered [3]. Here we investigate the continuity of the Julia set of the random iteration system and obtain

Theorem 5. *Let $J_w = J(w_1, w_2, \ldots, w_M)$ denote the Julia set of the random iteration system formed by $\mathcal{R}_w$. If $\mathcal{R}_a = \mathcal{R}(a_1, a_2, \ldots, a_M)$, $a \in \prod_{i=1}^M W_i$, is hyperbolic, the Julia set J_w is continuous at a.*

In this section, $W_\sigma(\mathcal{R}_w, z)$ is given by

$$W_\sigma^0(\mathcal{R}_w, z) = z, \qquad W_\sigma^n(\mathcal{R}_w, z) = R_{w_{j_n}} \circ R_{w_{j_{n-1}}} \circ \cdots \circ R_{w_{j_1}}(z), \quad n = 1, 2, \ldots,$$

for $\sigma = (j_1, j_2, \ldots, j_n, \ldots) \in \Sigma_M$.

To obtain our main result we need the following three lemmas.

$z \in \overline{\mathbf{C}}$ is called a repelling fixed point of $\mathcal{R}$ if there exist $\sigma \in \Sigma_M$ and a positive integer k such that $W_\sigma^k(z) = z$ and $|(W_\sigma^k)'(z)| > 1$. From [5], we have

Lemma 1. *All repelling fixed points of $\mathcal{R}$ are in the Julia set $J(\mathcal{R})$, and moreover they are dense in $J(\mathcal{R})$.*

Lemma 2. *Let U be a component of $F(\mathcal{R})$ and $g(z)$ be a limit function in U. If $g(\eta) \in J(\mathcal{R})$ for some $\eta \in U$, then $g(z)$ is constant in U.*

Proof. If the conclusion is false, the limit function $g(z)$ is non-constant in U, it is easy to see that $g(U) \subset F(\mathcal{R})$ and $g(\eta) \in F(\mathcal{R})$; this contradict our assumption. The lemma follows. $\square$

Lemma 3. *If $\mathcal{R}$ is hyperbolic, then there exist a compact K in $F(\mathcal{R})$ and a positive integer p such that for any $z \in F(\mathcal{R})$ and $\sigma \in \Sigma_M$, when $n > p$, we have $W_\sigma^n(z) \in K$.*

Proof. Suppose to the contrary that there are $b \in F(\mathcal{R})$, $\sigma_0 \in \Sigma_M$ and a sequence $\{n_i\}$ of positive integers with $\lim_{i\to\infty} n_i = +\infty$ such that $\lim_{i\to+\infty} W_{\sigma_0}^{n_i}(b) = \beta \in J(\mathcal{R})$. Since $b \in F(\mathcal{R})$, by passing to a subsequence of n_i if necessary, we assume further that $\lim_{i\to\infty} W_{\sigma_0}^{n_i}(z) = \psi(z)$ locally uniformly in the component of $F(\mathcal{R})$ containing b. Since $\psi(b) = \beta$, by Lemma 2, $\psi(z) \equiv \beta$ in U. Theorem 1 implies that $\beta \in P(\mathcal{R})$, which contradicts the fact that $\mathcal{R}$ is hyperbolic. The proof is complete. $\square$

Proof of Theorem 5. Since J_a is perfect [5], from Lemma 1, for any $\varepsilon > 0$, we may take a set of points in J_a, say $A = \{b_1, b_2, \ldots, b_l\}$, such that each element in A is a repelling fixed point of $\mathcal{R}_a$ and

$$[A]_{\varepsilon/2} \supset J_a,$$

where $a = (a_1, a_2, \ldots, a_M)$ and $[A]_\varepsilon$ denotes an ε-neighborhood of A. Implicit function theorem implies that there exists neighborhood $O_i \subset W_i$ of a_i, $i = 1, 2, \ldots, M$, such that when $w_i \in O_i$, there is some repelling fixed point b_i' of $\mathcal{R}(w_1, w_2, \ldots, w_M) = \{R_{w_1}, R_{w_2}, \ldots, R_{w_M}\}$ satisfying

$$|b_i - b_i'| \leqslant \varepsilon/2.$$

Let $A' = \{b_1', b_2', \ldots, b_l'\}$. Then $[A']_{\varepsilon/2} \supset A$. Let us assume $b = (w_1, w_2, \ldots, w_M) \in \prod_1^M O_i$. Hence $[J_b]_{\varepsilon/2} \supset A$ and

$$[J_b]_\varepsilon \supset J_a. \tag{2}$$

From Lemma 3 it is easily seen that for any compact subset K_0 of the Fatou set F_a of $\mathcal{R}_a$, the complement of J_a, we may take a neighborhood X_0 of a and a compact subset K in F_a with $K_0 \supset K$ such that for $b \in X_0$ and some integer $N_0 > 0$, as $n > N_0$,

$W_\sigma^n(\mathcal{R}_b, K_0) \subset K$, $\sigma \in \Sigma_M$. Now take an ε-neighborhood $[J_a]_\varepsilon$ of J_a. Then $Q_1 = \overline{\mathbf{C}} - [J_a]_\varepsilon$ is a compact subset of F_a. Hence there are a neighborhood X_1 of a and a compact subset K_1 in F_a such that for some integer N_1, when $n > N_1$ and $b \in X_1$, $W_\sigma^n(\mathcal{R}_b, Q_1) \subset K_1$, $\sigma \in \Sigma_M$. Thus Q_1 lies in F_b and

$$[J_a]_\varepsilon \supset J_b. \tag{3}$$

(2) and (3) show that J_w is continuous at a under the Hausdorff metric. □

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